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Simple Comparison of Convergence of General Iterations and Effect of Variation in Various Parameters

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***Abstract*—In this paper, we study the stability analysis of Jungck-Ishikawa and Jungck-Noor iteration procedures for a number of examples. The effect on the convergence with the variation of parameters in both iterative schemes is shown.**

***Index Terms*— Jungck -Noor iteration procedure, Jungck -Ishikawa iteration procedure, convergence, stability result, Fixed point iteration procedure.**

I. INTRODUCTION

Let (X, d) be a complete metric space and $T : X \to X$. Let F_T be the set of fixed point of T, that is, $F_T = \{x \in X : Tx = x\}$. The sequence of iterates $\{x_n\}_{n=0}^{\infty}$ determined by,

$$x_{n+1} = Tx_n, n = 0,1,2,... \quad (1.1)$$

It is called Picard iteration process. But in the case of slightly weaker contractive condition, this scheme may not converge to the fixed point. Mann [10] introduced a new iterative scheme as follows.

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, n = 0,1,... \quad (1.2)$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$. But Mann does not converge to a fixed point if T is not continuous. To overcome this difficulty, some other iteration schemes may be used.

If for $x_0 \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by,

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Tz_n, \\ z_n &= (1-\beta_n)x_n + \beta_n Tx_n, n = 0,1,... \end{aligned} \quad (1.3)$$

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where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are the real sequences in $[0,1]$. This is the Ishikawa iteration scheme [8].

If $\alpha_n = 1$ for all 'n' in (1.2), it becomes Picard iteration scheme (1.1). Similarly if $\beta_n = 0$ for each 'n' in (1.3), it reduces to Mann iteration (1.2).

M. A. Noor [11] defined the following scheme for $x_0 \in X$,

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, n = 0,1,... \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are the real sequences in $[0,1]$. On putting $\gamma_n = 0$ for each 'n', (1.4) becomes (1.3).

These iteration schemes have been extensively studied in the literature in the light of Jungck [9] contraction (see [3], [12, 13], [25] and reference thereof).

Let Y be an arbitrary non empty set and (X,d) a metric space. Let $S,T : Y \to X$ and $T(Y) \subset S(Y)$ for some $x_0 \in Y$, consider

$$Sx_{n+1} = f(T, x_n), n = 0,1,2... \tag{1.5}$$

for $Y = X$ and $f(T,x_n) = Tx_n$, the iterative procedure (1.5) yields the Junck iteration [9].

Singh et al. [26] defined Junck-Mann iteration process as follows.

$$Sx_{n+1} = (1-\alpha_n)Sx_n + \alpha_n Tx_n, n = 0,1,2... \tag{1.6}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0,1]$.

Olatinwo and Imoru [12] defined Jungck-Ishikawa iteration scheme as,

$$\begin{aligned} Sx_{n+1} &= (1-\alpha_n)Sx_n + \alpha_n T z_n, \\ Sz_n &= (1-\beta_n)Sx_n + \beta_n T x_n, n = 0,1,... \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are the real sequences in $[0,1]$.

Further it is extended by Olatinwo [13] in the following manner.

Let $S : X \to X$ and $T(X) \subseteq S(X)$. Define

$$\begin{aligned} Sx_{n+1} &= (1-\alpha_n)Sx_n + \alpha_n T z_n, \\ Sz_n &= (1-\beta_n)Sx_n + \beta_n T r_n, \\ Sr_n &= (1-\gamma_n)Sx_n + \gamma_n T x_n \end{aligned} \tag{1.8}$$

where $n = 0,1,...$ and $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ satisfies

(i) $\alpha_0 = 1$ (ii) $0 \le \alpha_n, \beta_n, \gamma_n \le 1, n > 0$

(iii) $\sum \alpha_n = \infty$ (iv) $\sum_{j=0}^{n} \alpha_j \prod_{i=j+1}^{n} (1-\alpha_i + a\alpha_i)$ converges.

This scheme is also called Jungck-Noor iteration scheme. On putting $Y = X$ and $S = id$, the identity map on X, (1.6) reduces to (1.2), (1.7) becomes (1.3) and (1.8) becomes (1.4).

These iterative procedures give a fruitful result only when the procedure is stable with respect to the map under consideration. The stability of iterative scheme was first defined by Harder and Hick [6, 7] as follows.

An iterative procedure $x_{n+1} = f(T, x_n)$ is said to be T–stable with respect to a mapping T if $\{x_n\}$ converges to a fixed point q of T and whenever $\{y_n\}$ is a sequence in X with $\lim_{n\to\infty} d(y_{n+1}, f(T, y_n)) = 0$, we have $\lim_{n\to\infty} y_n = q$.

Singh et al. [26] extended it for $(S-T)$ stability in a new setting. A number of authors studied and extended the notion of various iterative schemes in different settings see for instance Phuengrattana and Suantai [15], Rhoades and Soltuz [23-24], Berinde [1-3], Chugh et al. [4, 5].
In this paper, we are doing comparative study of rate of convergence of Jungck-Ishikawa and Jungck-Noor iterative procedures with the help of some examples. The variations of number of iterations (N) versus the values of α, β, γ and x_0 for Junck-Noor and Jungck-Ishikawa iteration are also studied. In this study we take $\alpha_n = \alpha, \beta_n = \beta$ and $\gamma_n = \gamma$ for all *n*.

II. Results and Discussions

Example2.1. Consider the non-linear equation $\sin(x) - (e^x - 2) = 0$. Let us take $Tx = \sin(x)$ and $Sx = (e^x - 2)$. If we choose initial guess $x_0 = 1.5$, then from the Table I, it is observed that Jungck-Ishikawa iteration scheme (for $\alpha = 0.9337912, \beta = 0.1$) and Jungck-Noor iteration scheme (for $\alpha = 0.9337912, \beta = 0.1$ and γ=0.2) computes the result in same number of iterations.

Table I. Convergence of Jungck-Ishikawa and Jungck-Noor Iterative Schemes

n	JI ($\alpha = 0.9337912, \beta = 0.1$)				n	JN ($\alpha = 0.9337912, \beta = 0.1$ and γ=0.2)			
	Tx_n	Sx_{n+1}	x_{n+1}	ε_n		Tx_n	Sx_{n+1}	x_{n+1}	ε_n
0	0.9975	1.0930	1.1291	0.0955	0	0.9975	1.0930	1.1291	0.0955
1	0.9040	0.9141	1.0696	0.0100	1	0.9040	0.9140	1.0695	0.0100
2	0.8770	0.8789	1.0574	0.0019	2	0.8770	0.8788	1.0574	0.0019
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
5	0.8695	0.8695	1.0541	0.0000	5	0.8695	0.8695	1.0541	0.0000

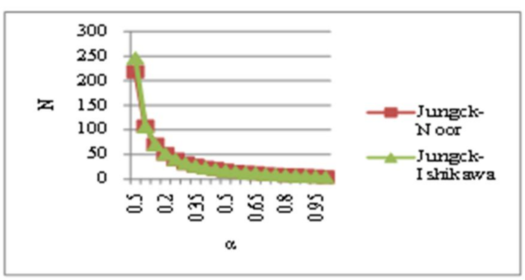

Figure 1. Effect of α *on no. of iteration*

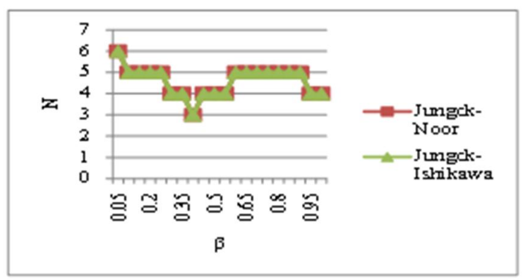

Figure 2. Effect of β on no. of iteration

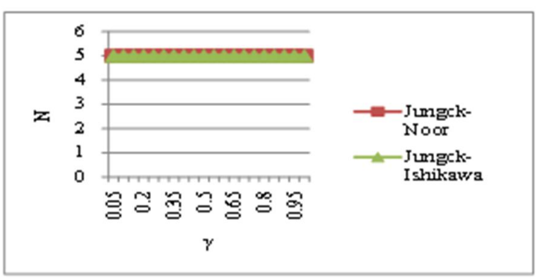

Figure 3. Effect of γ on no. of iteration

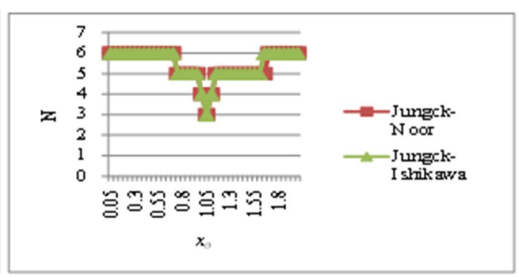

Figure 4. Effect of X_0 on no. of iteration

Example2.2. Consider the equation $e^{3x} - \tan x = 0$. Let us take $Tx = e^{3x}$ and $Sx = \tan x$. If we choose initial guess $x_0 = 1.0$, then Jungck-Ishikawa iteration scheme (for $\alpha = 0.7256, \beta = 0.312$) and Jungck-Noor iteration scheme (for $\alpha = 0.7256, \beta = 0.312$ and γ=0.278) computes the result in 12 iterations (Table II).

TABLE II. CONVERGENCE OF JUNGCK-ISHIKAWA AND JUNGCK-NOOR ITERATIVE SCHEMES

n	JI ($\alpha = 0.7256, \beta = 0.312$)				n	JN ($\alpha = 0.7256, \beta = 0.312$ and γ=0.278)			
	Tx_n	Sx_{n+1}	x_{n+1}	ε_n		Tx_n	Sx_{n+1}	x_{n+1}	ε_n
0	20.0855	54.2296	1.5524	34.1440	0	20.0855	71.4671	1.5568	51.3816
1	105.3275	92.2726	1.5600	13.0549	1	106.7420	97.5034	1.5605	9.2386
2	107.7569	103.6346	1.5611	4.1223	2	107.9450	105.1585	1.5613	2.7865
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
12	108.2759	108.2759	1.5616	0.0000	12	108.2759	108.2759	1.5616	0.0000

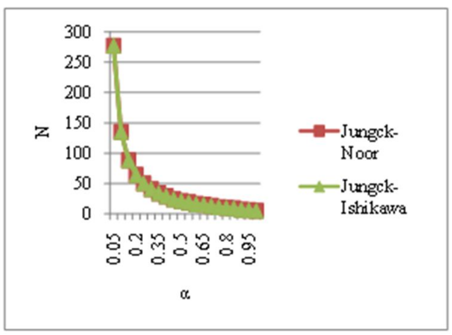

Figure 5. Effect of α on no. of iteration

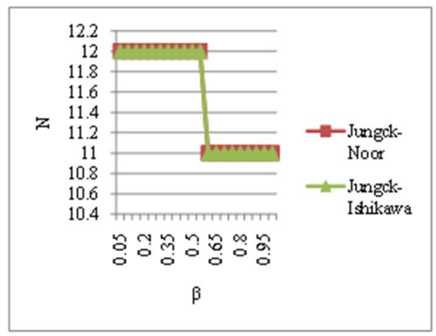

Figure 6. Effect of β on no. of iteration

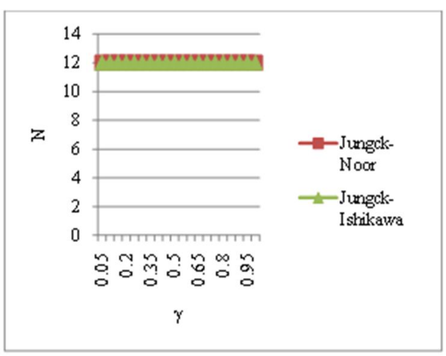

Figure 7. Effect of γ on no. of iteration

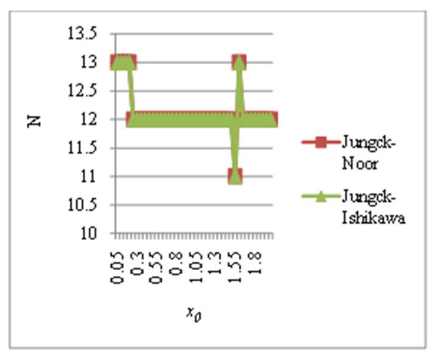

Figure 8. Effect of X_0 *on no. of iteration*

Example2.3. Consider the non-linear equation $e^{(\frac{2x+1}{4})^2} + x = 1$. Let us take $Tx = e^{(\frac{2x+1}{4})^2}$ and $Sx = (1-x)$. If we choose initial guess $x_0 = 1.5$, then from the Table III, Jungck-Ishikawa iteration scheme (for $\alpha = 0.8939, \beta = 0.125$) and Jungck-Noor iteration scheme (for $\alpha = 0.8939, \beta = 0.125$ and γ=0.327) computes the solution in 6 number of iterations.

TABLE III. CONVERGENCE OF JUNGCK-ISHIKAWA AND JUNGCK-NOOR ITERATIVE SCHEMES

n	JI ($\alpha = 0.8939, \beta = 0.125$)				n	JN ($\alpha = 0.8939, \beta = 0.125$ and γ=0.327)			
	Tx_n	Sx_{n+1}	x_{n+1}	ε_n		Tx_n	Sx_{n+1}	x_{n+1}	ε_n
0	2.7183	1.6391	-0.6391	1.0792	0	2.7183	1.9225	-0.9225	0.7957
1	1.0049	1.0686	-0.0686	0.0638	1	1.0456	1.1193	-0.1193	0.0737
2	1.0476	1.0504	-0.0504	0.0028	2	1.0369	1.0474	-0.0474	0.0105
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
6	1.0516	1.0516	-0.0516	0.0000	6	1.0516	1.0516	-0.0516	0.0000

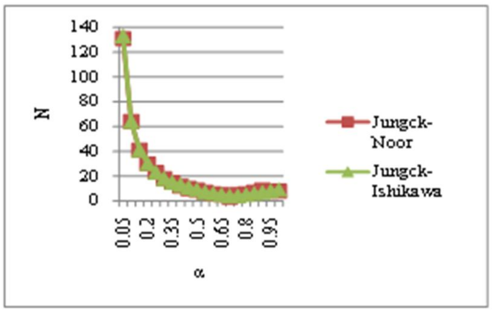

Figure 9. Effect of α on no. of iteration

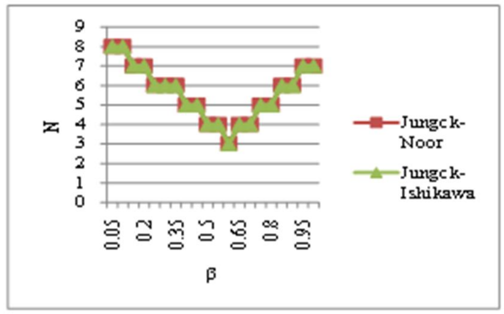

Figure 10. Effect of β on no. of iteration

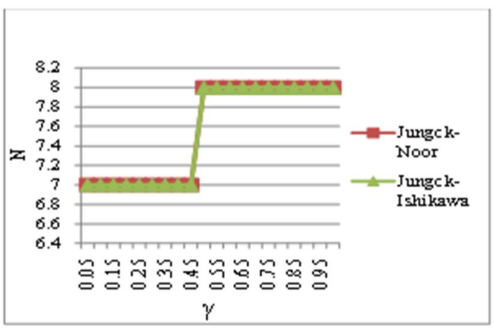

Figure 11. Effect of γ on no. of iteration

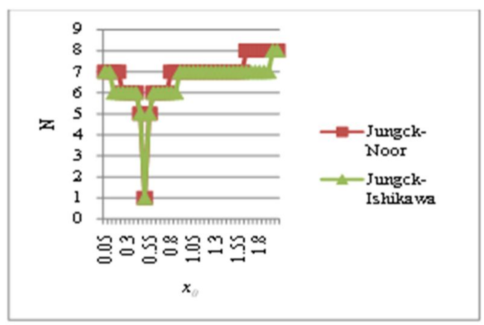

Figure 12. Effect of X_0 on no. of iteration

Example2.4. Consider the equation $e^x - 2 - \sin(e^x) = 0$. Let us take $Tx = \sin(e^x)$ and $Sx = e^x - 2$. If we choose initial guess $x_0 = 0.5$, then from the Table IV it is observed that Jungck-Ishikawa iteration scheme (for α=0.8934,β=0.453) and Jungck-Noor iteration (for α=0.8934,β=0.453,γ=0.127) evaluates the solution in same number of iterations.

TABLE IV. CONVERGENCE OF JUNGCK-ISHIKAWA AND JUNGCK-NOOR ITERATIVE SCHEMES

n	JI (α=0.8934,β=0.453)				n	JN (α=0.8934,β=0.453,γ=0.127)			
	Tx_n	Sx_{n+1}	x_{n+1}	ε_n		Tx_n	Sx_{n+1}	x_{n+1}	ε_n
0	0.9970	0.6523	0.9754	0.3446	0	0.9970	0.6594	0.9781	0.3375
1	0.4700	0.5531	0.9373	0.0831	1	0.4637	0.5456	0.9343	0.0819
2	0.5551	0.5542	0.9377	0.0009	2	0.5014	0.5549	0.9380	0.0064
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
5	0.5542	0.5542	0.9377	0.0000	5	0.5542	0.5542	0.9377	0.0000

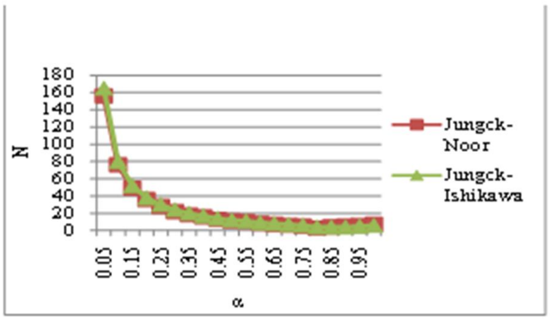

Figure 13. Effect of α on no. of iteration

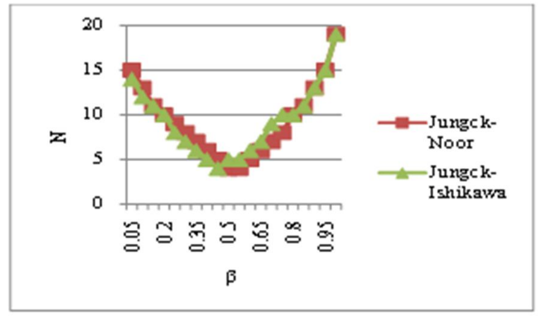

Figure 14. Effect of β on no. of iteration

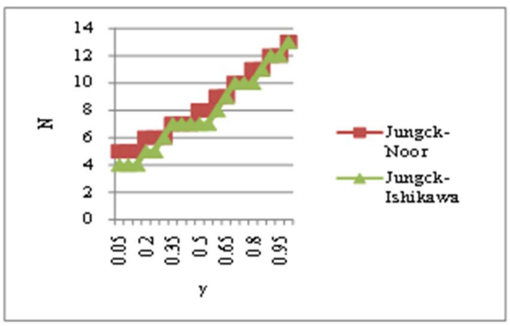

Figure 15. Effect of γ on no. of iteration

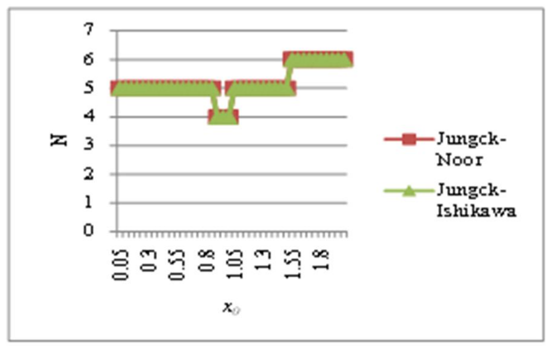

Figure 16. Effect of X_0 on no. of iteration

III. CONCLUSIONS

The variation in number of iterations with changing values of α, β, γ and x_0 is studied. It is very exciting to figure out that convergence rates of both the iterative procedures that is Jungck-Noor and Jungck-Ishikawa iterative procedures are almost similar with changing any factor (α, β, γ and x_0) at almost each and every step for the selected examples. Further, the following observations are noticed.

1. At constant values of β and γ, number of iterations falls rapidly initially with changing α but then starts to decrease gradually (Figs. 1.1, 2.1, 3.1, 4.1).
2. No specific pattern is observed while changing β at constant α and γ (Figs. 1.2, 2.2, 3.2, 4.2).
3. The number of iterations remains almost constant with changing the values of γ at constant α and β (Figs. 1.3, 2.3, 3.3, 4.3).
4. At a specific choice of x_0, the result converges in minimum number of iterations (Figs. 1.4, 2.4, 3.4, 4.4).

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